Energy landscape of 2D fluid foams

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Abstract: The equilibrium states of 2D non-coarsening fluid foams, which consist of bubbles with fixed areas, correspond to local minima of the total perimeter. (i) We find an approximate value of the global minimum, and determine directly from an image how far a foam's energy is from its ground state. (ii) For (small) area disorder, small bubbles tend to sort inwards and large bubbles outwards. (iii) Topological charges of the same sign 'repel' while charges of opposite sign 'attract'. (iv) We discuss boundary conditions and the uniqueness of the pattern for fixed topology.

1 Introduction

A 2D fluid foam or cellular fluid consists of a collection of bubbles separated by a continuous phase which tends to minimize its perimeter energy for fixed bubble areas.

Mathematically, what is the minimal perimeter enclosing a cluster of N bubbles with known areas [1]? We estimate the value of the perimeter minimum, and conjecture the corresponding patterns, for cases that have thus far escaped rigorous study, including large N, real boundary conditions and area dispersity. We hope to provide insight for future rigorous mathematical proofs. Physically, most studies on foam structure have focused on special consequences of the energy minimization [2, 3], such as topology. Here we derive more general consequences, to address open questions such as: Can we determine on an image whether the foam is stressed and deformed? How do topology (the number of neighbors of a bubble), pressure and energy relate? Why do pentagons and heptagons tend to cluster in pairs in 2D foams? More practically, understanding a foam's energy lansdcape is an important step towards predicting the quasistatic stress-strain relationship.

To capture the essential features of energy minimization, we make the following restrictions, which we can relax later. (i) 2D foam. (ii) "Dry foam" limit: fluid fraction $\phi \ll 1$. (iii) Weakly disordered foams "close to a honeycomb structure:" both the area and the edge number distributions have small variance. (iv) Each bubble's area is constant. In fact, after a mechanical perturbation, bubble walls equilibrate on a much shorter time-scale than area changes due *e.g.* to coarsening, wall breakage, cell division, and cell nucleation.

For N bubbles with given areas $\{A_{i=1,\ldots,N}\}$ and a line tension γ, the foam energy is simply the sum of edge lengths ℓ_{ij} between all pairs of neighboring bubbles i and j ($i = 0$ denotes the outer fluid):

$$H = \gamma \sum_{0 \le i < j \le N} \ell_{ij}. \tag{1}$$

Figure 1: *Simulated foams with fixed areas: (a) A typical configuration of a polydispersed foam at equilibrium. The top and bottom boundaries are fixed, the lateral boundaries periodic. Shades of grey encode the topology. (b) A pentagon-heptagon-pentagon-heptagon cluster artificially constructed in a regular foam with equal areas and periodic boundaries. (c) Two dipoles (pentagon-heptagon pairs) result in a curvature field in the hexagons around them. (d) A circular obstacle in the center of a hexagonal foam induces a topological charge distribution.*

At equilibrium, *i.e.* in a local energy minimum, the foam obeys the Plateau rules [2, 3]: bubble edges are circular arcs which meet in triples at $2\pi/3$ angles [4]. According to Laplace's law their algebraic curvatures ($\kappa_{ij} = -\kappa_{ji} > 0$ when bubble i is convex compared with bubble j) relate to the 2D pressure P_i inside bubble i: $\kappa_{ij} = \frac{P_i - P_j}{\gamma}$. Thus the algebraic curvatures of the three edges that meet at the same vertex add to zero [5]:

$$\kappa_{ij} + \kappa_{jk} + \kappa_{ki} = 0. \tag{2}$$

Eq. (2) extends to any closed contour crossing more edges.

2 Zeroth-order estimate of the global minimum

If all the bubble areas A_i are known but their topology is free to vary, the minimum value (the ground state) for the foam energy exists [6]. However, we do not know how to determine its value or the corresponding pattern(s). We conjecture [7] that the energy of a natural, random foam is at least the energy of a collection of regular hexagons with the same areas A_i (*i.e.* perimeter $3.72\,\sqrt{A_i}$):

$$min(H) \geq H_h = 3.72\,\frac{\gamma}{2}\sum_{i=1}^{N}\sqrt{A_i}. \tag{3}$$

This estimate H_h of the global minimum depends only on the area distribution, not the pattern. Thus given an image, we can simultaneously measure H, through the actual edge lengths, and H_h, through the areas. The ratio H/H_h is a global marker of the energy stored in the foam, or how far the foam is from its global minimum at prescribed areas.

We use the extended large-Q Potts model, which allows large numbers of bubbles, $N \gg 1$, no fluid fraction $\phi = 0$, fixed bubble areas, a large range of area distributions and quick equilibration [8] (Fig. 1a). By biasing the Monte Carlo process, we can apply a steady shear [8] to prepare a distorted foam: a higher shear rate results in more distorted bubbles, thus higher initial energy H_i. We let the distorted foams relax towards equilibrium, *i.e.* towards a local energy minimum. Fig. (2) shows that whatever their initial energies, the relaxed foams all have final energies H_f 2% above H_h. We now write the energy of a nearly regular foam as the ground state H_h plus three corrections.

3 Area disorder

A regular hexagon of area A has edge lengths $L = 3.72\sqrt{A}/6$. If two bubbles of different areas $A_i > A_j$ share a common edge, its length ℓ_{ij} obviously cannot be simultaneously

Figure 2: *Relaxed energy of a random foam (Fig. 1a). The final rescaled equilibrium energies, H_f/H_h, after long relaxations (10^6 MCS) are plotted against their initial energy, H_i/H_h. Note the difference in horizontal and vertical scales.*

equal to both L_i and L_j. The area mismatch, $\epsilon = (A_i - A_j)/(A_i + A_j)$, costs an energy which vanishes only for $\epsilon = 0$ [7]. If the area disorder is small enough [9], the foam reduces its energy when the topological disorder is small: each bubble is surrounded by neighbors of nearly the same areas and reaches a nearly regular shape, *i.e. bubbles sort according to their sizes*. With free boundaries and no external force field, we expect smaller bubbles to sort inwards, larger ones outwards.

Figure 3: *Photograph of a ferrofluid foam. The heptagon (7) has two pentagonal neighbours.*

We observe sorting in an "annealed" ferrofluid foam prepared between two Plexiglas plates [10]. We tilt the plates from the horizontal plane to an angle of $0.1°$, inducing a low effective gravity field. Large bubbles drift upwards, small bubbles downwards, resulting in vertical sorting according to size [11]. We then bring the plates back to horizontal, and the bubbles slowly drift back and settle. The bubbles rearrange and explore the energy space to find a stable lower energy configuration (Fig. 3) with rounded bubbles and radial sorting according to size, larger bubbles surrounding smaller ones.

4 Topological disorder and electrostatic analogy

If the fluid fraction ϕ is small enough that we can interpolate P within the Plateau borders, the gradient of pressure, $-\vec{\nabla} P$ is proportional to edge curvatures (Laplace's law) and its circulation along any closed contour is zero. Let us relate it to the "topological charge" which quantifies the deviation from a hexagonal lattice: an n-sided bubble has a charge $q = (6 - n)$, *i.e.* a dislocation $(6 - n)\pi/3$.

Extending the *topological* Gauss-Bonnet theorem [12], we have demonstrated the following *geometrical* Gauss-Bonnet theorem [7]. If a closed contour C (with $\hat{n}$ its normal) encloses bubbles with charges q_i, then the outwards flux of $-\vec{\nabla} P$ across C is determined by the total charge $Q = \sum q_i$:

$$\oint_C \vec{\nabla} P \cdot \hat{n} \, d\ell \propto Q, \tag{4}$$

in analogy with 2D electrostatics.

The analogy (Table I) includes topological "dipoles", "quadrupoles" and the distortion they induce in a regular lattice (Fig. 1b, 1c). It even extends to the expression of energy [7]. An isolated topological charge is rare in real foam, because its energy cost diverges (logarithmically) with the foam size. Conversely, two charges of opposite sign lower their energy when they approach, hence the "effective attractive interaction" [2] invoked to explain the frequency of heptagon-pentagon pairs (Fig. 3).

5 Boundary conditions

The possible boundary conditions for foam are: periodic, Fig. (1b); free, if the foam is surrounded by a fluid medium, Fig. (3); fixed, if it touches a solid box, Fig. (1d); or some combination of these three, Fig. (1a).

For a free foam, the outer fluid fixes the pressures at the foam boundary. A fixed boundary requires that the pressure gradient $-\vec{\nabla}P$ (perpendicular to each bubble edge, itself perpendicular to the boundary) is parallel to the boundary. Thus in both cases, the topological charges should *uniquely* determine the pressure in the foam as a Dirichlet or a Neumann problem, respectively [7].

Periodic boundary conditions guarantee that the total charge $Q = 0$ (Euler theorem). For all other boundary conditions, we can apply the topological Gauss-Bonnet theorem [12] to the N_b bubbles at the foam's boundary: the total charge of a foam is $Q = (N_b + 6)$. Introducing a modified definition $\tilde{q} = (5 - n)$ instead of $(6 - n)$ for the N_b bubbles at the boundary is convenient: the total topological charge of a foam becomes simply $\tilde{Q} = \sum_i \tilde{q}_i = 6$ (*i.e.* a dislocation of $6\pi/3 = 2\pi$).

The shape of the bounding box (or the concave boundary of an obstacle placed in the middle of the foam, Fig. 1d) determines the distribution of these 2π (-2π) among the bubbles touching the boundary. If boundary has all corner angles a multiple of $\pi/3$, all bubble edges can simultaneously be straight (isobaric foam). In general, however, the boundary results in curved bubble edges.

6 Summary

The equilibrium energy of 2D foams with given bubble areas helps explain foam structure. The zeroth-order estimate of the ground energy, H_h, is a function of the area distribution only, independent of the topology. The ratio H/H_h provides a global marker of how far a foam is from its ground state. We have similarly defined a local marker for each edge length, allowing an intrinsic definition of strain [7].

The present analysis, independent on the characteristic size and energy scales, physically explains the different contributions to a foam's energy: area mismatch, topology and boundaries. It predicts the foam configurations corresponding to ground states and is valid for foams in which bubble areas vary slowly.

The pressure, edge curvature and topological charge form a set of good variables to characterize a foam. They present a profound analogy with 2D electrostatic potential, field and charge, respectively, through the geometrical Gauss-Bonnet theorem (eq. 4). The topology determines a foam's energy and pattern and explains the origin of topological and geometrical correlations. Bubbles sort according to size. Topological charges of same sign repel and of opposite sign attract.

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	potential	field	charge
$2D$ electrostatics	potential V	electric field $-\vec{\nabla}V$	electric charge e
$2D$ foams	pressure P	curvature $\propto -\vec{\nabla}P$	topological charge $\propto (6-n)$

Table 1: Proposed analogy between foams and electrostatics in two dimensions.

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